A theorem on error analysis

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Abstract

In this paper on error analysis we come back to definitions in such a way that we will be able to give formal aspect to the topic. This attempt is based upon a method on error analysis, as taught to students in my university (USTO), called in French (language of teaching in Algeria): *Méthode de la Différentielle Logarithmique* (MDL). We translate to English as: *Method of Differential Logarithm* (MDL). Following this method step by step, we obtain a theorem on error analysis; may be this theorem has always been there but was often masked by concrete applications rather than formal aspects of problems.

What is MDL

Let G a given quantity. This quantity can be measured (calculated) indirectly via other independent quantities $\{g_1, g_2, g_3, \dots, g_k, \dots g_N\}$. We can then write:

$$G = f(g_1, g_2, g_3, \dots, g_k, \dots g_N) \qquad (a)$$

G is a function on $\{g_1, g_2, g_3, \dots, g_k, \dots g_N\}$

The **MDL**, as taught to students in USTO (Algeria), consists in the following steps:

1- Taking the logarithm $\ln$ of both sides of equation (a) above:

$$\ln G = \ln f(g_1, g_2, g_3, \dots, g_k, \dots g_N) \qquad (b)$$

The right hand side of (a) is often a product, division, power or even an exponential of $\{g_1, g_2, g_3, \dots, g_k, \dots g_N\}$

Applying $\ln$ operation on these terms as:

$$\ln (g_1 \, x \, g_2) = \ln g_1 + \ln g_2$$

$$\ln \frac{g_1}{g_2} = \ln g_1 - \ln g_2$$

$$\ln g_k^{\alpha} = \alpha \ln g_k$$

$$\ln e^{g_k} = g_k$$

Each of these equalities is a particular case of the following more general equality:

$$\ln G = \sum_k a_k \ln g_k + \sum b_l \, g_l \qquad (c)$$

Where the coefficients a_k and b_l depend on $\{g_1, g_2, g_3, \dots, g_k, \dots g_N\}$.

2 – Differentiating both sides of (c) we have:

$$d\ln G = \sum_k da_k \ln g_k + \sum_k a_k \, d\ln g_k + \sum_l db_l \, g_l + \sum_l b_l \, dg_l \qquad (c)$$

By definition we have:

$$df(x) = f'(x)\,dx$$

$$d\ln f = \frac{df}{f}$$

Thus (c) becomes :

$$\frac{dG}{G} = \sum_k da_k \ln g_k + \sum_k a_k \frac{dg_k}{g_k} + \sum_l db_l \, g_l + \sum_l b_l \, dg_l \qquad (d)$$

a_k and b_l are functions on $\{g_1, g_2, g_3, \ldots, g_k, \ldots g_N\}$, we have :

$$da_k = \sum_i \frac{\partial a_k}{\partial g_i} dg_i = \sum_i a_k'^{(i)} dg_i \quad , \quad db_l = \sum_j \frac{\partial b_l}{\partial g_j} dg_j = \sum b_l'^{(j)} dg_j$$

In order to make formulae easy to manipulate, we put:

$$a_k'^{(i)} = \frac{\partial a_k}{\partial g_i} = \sum_i dg_i \quad , \quad b_l'^{(j)} = \frac{\partial b_l}{\partial g_j}$$

Thus (d) becomes:

$$\frac{dG}{G} = \sum_k [\sum_i a_k'^{(i)} \, dg_i \,] \ln g_k + \sum_k a_k \frac{dg_k}{g_k} + \sum_l [\sum_j b_l'^{(j)} \, dg_j \,] g_l + \sum_l b_l \, dg_l \qquad (e)$$

The theorem

Rearranging all terms in the right hand side of (e) , it follows:

$$\frac{dG}{G} = \sum_m T_m dg_m + \sum M_n \frac{dg_n}{g_n} \qquad (f)$$

Where the coefficients T_m and M_n are functions on $\{g_1, g_2, g_3, \ldots, g_k, \ldots g_N\}$
One can prove (f) for a quantity G that depend only on two independent quantities $\{g_1, g_2\}$, that is for $G = f(g_1, g_2)$.

Equation (f) can be stated as a theorem on error analysis a follows: *The fractional error* $\frac{dG}{G}$ *of physical quantity* G *which depends on other independent quantities* $\{g_1, g_2, g_3, \ldots, g_k, \ldots g_N\}$ *is a linear combination of absolute* dg_k *and fractional* $\frac{dg_k}{g_k}$ *errors of these independent quantities.*

This statement (theorem) has always been there in any application of error analysis however was masked by concrete consideration rather than formal considerations. Even more, this statement (theorem) may be was proved by other methods. The original idea in the present proof is equation (c) which generalizes possible cases that we encounter in problems of error analysis.